

Integration by parts of some non-adapted vector field from Malliavin's lifting approach

Zhehua Li

February 23, 2017

Abstract

In this paper we propose a lift of vector field X on a Riemannian manifold M to a vector field \tilde{X} on the curved Cameron-Martin space $H(M)$ named orthogonal lift. The construction of this lift is based on a least square spirit with respect to a metric on $H(M)$ reflecting the damping effect of Ricci curvature. Its stochastic extension gives rise to a non-adapted Cameron-Martin vector field on $W_o(M)$. In particular, if $M = \mathbb{R}^d$ with Euclidean metric, then the damp disappears and the lift reduces to the well-known Malliavin's lift. We establish an integration by parts formula for these first order differential operators.

Contents

1	Introduction	1
1.1	Differential Structure on Path Spaces	1
1.2	Riemannian Metrics on $H(M)$ and Lifting Technique	3
1.3	Main Theorems	4
1.4	Structure of the Paper	5
2	Preliminaries in Geometry and Probability	5
3	The Orthogonal Lift \tilde{X} of X on $H(M)$ and Its Stochastic Extension	8
3.1	Damped Metrics and Adjoints	8
3.2	The Orthogonal Lift \tilde{X} on $H(M)$	10
4	A Differential Calculus on $W_o(M)$ for \tilde{X}	12
4.1	Review of Calculus on Wiener Space	12
4.2	Computing $\tilde{X}^{tr,\nu}$	15
A	ODE estimates	20
B	A Structure Theorem for $\operatorname{div}_g(\tilde{X})$	20

1 Introduction

1.1 Differential Structure on Path Spaces

Throughout this paper, we fix (M^d, g, ∇, o) to be a pointed complete Riemannian manifold of dimension d with Riemannian metric g , Levi-Civita covariant derivative (∇) , and base point $o \in M$. We further let

$$W_o(M) := \{\sigma \in C([0, 1] \mapsto M) \mid \sigma(0) = o\}$$

be the **Wiener space** on M and let ν be the **Wiener measure** on $W_o(M)$ —i.e. the law of M -valued Brownian motion which starts at $o \in M$. In order to highlight the effect of curvature in our

paper we reserve the symbol $(W_0(\mathbb{R}^d), \mu)$ for the Wiener space and Wiener measure on $W_0(\mathbb{R}^d)$ and refer to this pair as the classical Wiener space. In contrast, $(W_o(M), \nu)$ is usually referred to as curved Wiener space.

Differential calculus on $W_o(M)$ which is compatible with ν has been extensively explored and has been the main tool of modern stochastic analysis. The first question in this direction is to specify a differential structure (tangent space \mathcal{X} of the path space) that is compatible with Wiener measure ν , i.e. for any vector field (first order differential operator) $Y \in \mathcal{X}$, we can find an “integral curve” or “flow” ϕ_t at least in probability, such that

$$Y(\phi_t) = \frac{d}{dt}\phi_t \text{ or } Yf(\phi_t) = \frac{d}{dt}f(\phi_t) \text{ for some } \nu - \text{measurable function } f.$$

A minimum requirement to achieve the above result is the well-definedness of $f(\phi_t)$, i.e. the law of $\phi_t : W_o(M) \rightarrow W_o(M)$ should be equivalent to ν . Cameron and Martin [4] first proposed a differential structure named Cameron-Martin space which is further developed as the most natural tangent space on abstract Wiener space, see Theorem 1.2.

Definition 1.1 (Cameron-Martin space) *Let*

$$H(\mathbb{R}^d) := \left\{ \sigma \in C([0, 1] \mapsto \mathbb{R}^d) : \sigma(0) = 0, \sigma \text{ is a.c. and } \int_0^1 |\sigma'(s)|^2 ds < \infty \right\}$$

*be the **Cameron-Martin space** on \mathbb{R}^d . (Here a.c. means absolutely continuous.)*

Theorem 1.2 (Cameron-Martin) *For any $h \in H(\mathbb{R}^d)$, consider the flow ϕ_t^h generated by h , i.e. for any $w \in W_0(\mathbb{R}^d)$, $\phi_t^h(w) = w + th$. Notice that ϕ_t^h is the flow of the vector field $D_h := \frac{\partial}{\partial h}$. Then the pull-back measure $\mu^h(\cdot) := (\phi_1^h)_* \mu(\cdot) = \mu(\cdot - h)$ and Wiener measure μ are equivalent.*

The map ϕ_t^h is usually called Cameron-Martin shift and the phenomenon described in Theorem 1.2 is called quasi-invariance of μ under the Cameron-Martin shift. The generalization of Cameron-Martin Theorem to curved Wiener space came quite a while later in 1990s. Driver initiated the geometric Cameron-Martin theory in [8] and [9] where he considered a Cameron-Martin vector field X^h (see Definition 3.13) in which $h \in \{f \in C^1([0, 1]) : f(0) = 0\} \subset H(\mathbb{R}^d)$.

Theorem 1.3 (Driver) *Let (M, g, o, ∇) be a compact manifold and h be as above, then for any $\sigma \in W_o(M)$, there exists a unique flow ϕ_t^h of X^h , i.e. $\phi_t^h : W_o(M) \mapsto W_o(M)$ satisfying:*

$$\frac{d}{dt}\phi_t^h(\sigma) = X^h(\phi_t^h(\sigma)) \text{ with } \phi_0^h = I$$

and $\nu_t^h(\cdot) := (\phi_t^h)_ \nu$ is equivalent to ν .*

The existence of the flow and the quasi-invariance of Wiener measure under this flow were later extended to Cameron-Martin vector field X^h with $h \in H(\mathbb{R}^d)$ in [15] and [14] and then to a geometrically and stochastically complete Riemannian manifold in [16] and [18]. Meanwhile certain flaws of these Cameron-Martin vector fields also arise. For example, it has been known that this space of vector fields does not form a Lie Algebra, see [6] and [1], and also the Itô map fails to be a diffeomorphism from $W_0(\mathbb{R}^d)$ to $W_o(M)$. Motivated by these issues, Driver introduced more general Cameron-Martin vector field in [11], see also [6], where h admits some randomness. It has been known that if h is certain adapted Brownian semi-martingale, see Definition 4.1, then a quasi-invariant flow can be constructed on $(W_0(\mathbb{R}^d), \mu)$ and with the help of Itô map, an approximate flow (not a real flow) can be constructed to define X^h on $(W_o(M), \nu)$. In this paper we consider a class of non-adapted Cameron-Martin vector field on $W_o(M)$, see Definition 3.19. The reason to study these vector fields is that they naturally arise from Malliavin’s lifting approach applied to a curved Wiener space where damp is considered. Since Malliavin’s lifting approach is the key tool of stochastic analysis in the study of hypo-elliptic differential operators, see [3], and damping effect naturally appears because of non-trivial curvature, see [5] and [13], it should be useful to study these non-adapted Cameron-Martin vector fields.

1.2 Riemannian Metrics on $H(M)$ and Lifting Technique

In this section we introduce the Cameron-Martin space on (M, o) which is a sub-manifold of $W_o(M)$. Its importance are twofold: First, the differential structure on $W_o(M)$, i.e. Cameron-Martin vector field (see Definition 3.13) can be viewed as a stochastic extension of the differential structure on $H(M)$. Secondly, Riemannian metrics on $H(M)$ give rise to a technique that allows us to lift a vector field from M to $W_o(M)$.

Definition 1.4 (Cameron-Martin space on (M, o)) Let

$$H(M) := \left\{ \sigma \in C([0, 1] \mapsto M) : \sigma(0) = o, \sigma \text{ is a.c. and } \int_0^1 |\sigma'(s)|_g^2 ds < \infty \right\}$$

be the **Cameron-Martin space** on (M, o) . (Here a.c. means absolutely continuous.)

Notation 1.5 Let $\Gamma(TM)$ be differentiable sections of TM and $\Gamma_\sigma(TM)$ be differentiable sections of TM along $\sigma \in H(M)$.

The space, $H(M)$, is an infinite dimensional Hilbert manifold which is a central object in problems related to the calculus of variations on $W_o(M)$. Klingenberg [19] contains a good exposition of the manifold of paths. In particular, Theorem 1.2.9 in [19] presents the differentiable structure of $H(M)$ in terms of atlases. For our purpose, it suffices to just specify its tangent bundle $TH(M)$ and its Riemannian metrics. In this paper we define two metrics on $H(M)$. G^1 -metric seems to be a natural metric to geometers, however a damped metric $\langle \cdot, \cdot \rangle_{Ric}$ involving Ricci curvature is more widely seen in the literature of stochastic geometry as a way to represent the damping effect of curvature.

Definition 1.6 (G^1 -metric) For any $\sigma \in H(M)$ and $X, Y \in \Gamma_\sigma^{a.c.}(TM)$, We define a metric G^1 as follows:

$$\langle X, Y \rangle_{G^1} = \int_0^1 \left\langle \frac{\nabla X}{ds}(s), \frac{\nabla Y}{ds}(s) \right\rangle_g ds,$$

where $\Gamma_\sigma^{a.c.}(TM)$ is the set of absolutely continuous vector fields along σ with finite energy, i.e. $\int_0^1 \left\langle \frac{\nabla X}{ds}(s), \frac{\nabla X}{ds}(s) \right\rangle_g ds < \infty$.

Remark 1.7 To see that G^1 is a metric on $H(M)$, we identify the tangent space $T_\sigma H(M)$ with $\Gamma_\sigma^{a.c.}(TM)$. To motivate this identification, consider a differentiable one-parameter family of curves σ_t in $H(M)$ such that $\sigma_0 = \sigma$. By definition of tangent vector, $\frac{d}{dt} \big|_0 \sigma_t(s)$ should be viewed as a tangent vector at σ . This is actually the case, for detailed proof, see Theorem 1.3.1 in [19].

Definition 1.8 Let $\langle \cdot, \cdot \rangle_{Ric}$ be the **damped metric** on $TH(M)$ defined by

$$\langle X, Y \rangle_{Ric} := \int_0^1 \left\langle \left[\frac{\nabla}{ds} + \frac{1}{2} Ric \right] X(s), \left[\frac{\nabla}{ds} + \frac{1}{2} Ric \right] Y(s) \right\rangle_g ds \quad (1.1)$$

for all $X, Y \in \Gamma_\sigma^{a.c.}(TM) = T_\sigma H(M)$ and $\sigma \in H(M)$. Here Ric is the Ricci tensor, see Notation 2.21.

Remark 1.9 A damped metric or connection naturally appears when a manifold is involved in order to illustrate the damping effect that comes from the curvature. Other than the literature mentioned at the end of Section 1.1, in another paper of the author [arxiv], we find an interesting phenomenon that if one discretizes $(H(M), G^1)$ by considering a class of piecewise geodesic space $H_{\mathcal{P}}(M)$ adapted to a partition \mathcal{P} of time with a metric $G_{\mathcal{P}}^1$ which is the Riemann sum approximation to the G^1 metric, then the orthogonal lift with respect to $G_{\mathcal{P}}^1$ -metric as well as its adjoint converges to those of the orthogonal lift with respect to the damped metric on $W_o(M)$.

In the category of differential geometry lifting approach is fairly concise to state. Given two differentiable manifold N, M and a submersion $F : N \rightarrow M$, for any differentiable function f on M , its lift \tilde{f} with respect to F is simply defined to be $f \circ F$ and for any $X \in \Gamma(TM)$, $\tilde{X} \in \Gamma(TN)$ is called a **lift** of X iff $F_*\tilde{X} = X$. Since F is a submersion, the existence of \tilde{X} is trivial but one should not expect uniqueness. Based on simple definition one can obtain

$$\tilde{X}\tilde{f} = \widetilde{Xf}.$$

On $(W_o(M), \nu)$ one would pursue the above formula in an average sense, i.e.

$$\mathbb{E}_\nu [\tilde{X}\tilde{f}] = \mathbb{E}_\nu [\widetilde{Xf}] \quad \forall f \in C_b^1(M). \quad (1.2)$$

In this paper we found a lift named orthogonal lift on $W_o(M)$ with respect to the end point evaluation map E_1 in the following way: first we establish a unique lift of a vector field $X \in \Gamma(TM)$ to $\Gamma(TH(M))$ by requiring it to have minimum norm induced from the damped metric defined in Definition 1.8, then a Cameron-Martin vector field is obtained by stochastic extension.

Since the orthogonal lift \tilde{X} is a non-adapted vector field on the curved Wiener space, it is not clear whether \tilde{X} is in the domain of the divergence operator on $W_o(M)$ or not. To the author's knowledge, even the characterization of the domain of the divergence operator on $W_0(\mathbb{R}^d)$ is not quite satisfactory. Therefore in this paper we adopt a weaker notion of differentiability than the well-known H -derivative. However it will be shown that it is enough to derive an integration by parts formula.

1.3 Main Theorems

In this section we state the main results of this paper while avoiding many technical details.

Lemma 1.10 *Let $E_1 : W_o(M) \rightarrow M$ be the **End point evaluation map**, i.e. $\forall \sigma \in W_o(M)$, $E_1(\sigma) = \sigma(1)$, then $E_1|_{H(M)}$ is a submersion.*

Proof. Since M is complete, for any $x \in M$, there exists a geodesic $\sigma \in H(M)$ such that $\sigma(0) = o$ and $\sigma(1) = x$. So $E_1|_{H(M)}$ is surjective. Then for any $\sigma \in H(M)$ and $v \in T_{\sigma(1)}M$, set $h(s) = su^{-1}(1)v$, $0 \leq s \leq 1$ and it is trivial to check $X^h(\sigma, \cdot) \in T_\sigma(H(M))$ and $(E_1|_{H(M)})_{\sigma,*}(X^h) = v$. So $(E_1|_{H(M)})_{\sigma,*}$ is surjective and thus $E_1|_{H(M)}$ is a submersion. (I am sorry for using some notations that have not been set up. Here $u(\sigma, \cdot)$ is the parallel translation along σ , see Definition 2.5 and X^h is defined in Notation 2.25). ■

Theorem 1.11 (Orthogonal Lift on $H(M)$) *If M has non-positive and bounded sectional curvature, then for any $X \in \Gamma(TM)$, there is a unique $\tilde{X} \in \Gamma(TH(M))$ such that for any $\sigma \in H(M)$,*

$$\|\tilde{X}(\sigma)\|_{Ric} = \inf \{\|Y(\sigma)\|_{Ric} : E_{1*}Y = X\}. \quad (1.3)$$

where $\|\cdot\|_{Ric}$ is the norm on $T_\sigma H(M)$ induced by the damped metric in Definition 1.8.

If we further consider its stochastic extension to $W_o(M)$, we get a non-adapted Cameron-Martin vector field (Still denoted by \tilde{X}), then we can prove:

Theorem 1.12 *Denote by $\mathcal{D}(\tilde{X})$ the domain of \tilde{X} which is dense on $L^2(W_o(M), \nu)$, then for any $f, g \in \mathcal{D}(\tilde{X})$, we have*

$$\mathbb{E}_\nu [\tilde{X}f \cdot g] = \mathbb{E}_\nu [f \cdot \tilde{X}^\dagger g] \quad (1.4)$$

where \tilde{X}^\dagger is a densely defined operator on $L^2(W_o(M), \nu)$ explicitly given in Lemma 4.23.

1.4 Structure of the Paper

For the guidance to the reader, we give a brief summary of the contents of this paper.

In Section 2 we set up some notations and preliminaries in probability and geometry. In particular we present the stochastic parallel translation which leads to the stochastic extension of \tilde{X} mentioned in Theorem 1.11 to $W_o(M)$.

In Section 3 we prove Theorem 1.11 in a constructive way and derive its stochastic extension accordingly.

In Section 4 we first explore the possibility of fitting \tilde{X} into existent theory by summarizing some classical results in differential calculus on $W_o(M)$. Some difficulties are mentioned in this direction. Then we set up a differential calculus for \tilde{X} on $W_o(M)$ and derive an integration by parts formula for it. In the last of this section we explore the divergence term of the adjoint of \tilde{X} under the condition that the curvature tensor is parallel.

Acknowledgement 1.13 *I want to thank my advisor Bruce Driver for introducing to me Malliavin's lifting approach, especially its non-adapted nature, in contrast to Bismut's adapted lifting approach, both are powerful tools in Stochastic analysis.*

2 Preliminaries in Geometry and Probability

For the remainder of this paper, let $u_0 : \mathbb{R}^d \rightarrow T_o M$ be a fixed linear isometry which we add to the standard setup (M, g, o, u_0, ∇) . We use u_0 to identify $T_o M$ with \mathbb{R}^d . Suggested references for this section are Section 2 of [17] and Sections 2, 3 of [8]. Some other references are [2], [12], [6] and [10] to name just a few.

Definition 2.1 (Orthonormal Frame Bundle $(\mathcal{O}(M), \pi)$) *For any $x \in M$, denote by $\mathcal{O}(M)_x$ the space of orthonormal frames on $T_x M$, i.e. the space of linear isometries from \mathbb{R}^d to $T_x M$. Denote $\mathcal{O}(M) := \cup_{x \in M} \mathcal{O}(M)_x$ and let $\pi : \mathcal{O}(M) \rightarrow M$ be the (fiber) projection map, i.e. for each $u \in \mathcal{O}(M)_x$, $\pi(u) = x$. The pair $(\mathcal{O}(M), \pi)$ is the orthonormal frame bundle over M .*

Definition 2.2 (Connection on $\mathcal{O}(M)$) *The connection on $\mathcal{O}(M)$ used in this paper is uniquely specified by the $\mathfrak{so}(d)$ -valued connection form ω^∇ on $\mathcal{O}(M)$ determined by ∇ ; for any $u \in \mathcal{O}(M)$ and $X \in T_u \mathcal{O}(M)$,*

$$\omega_u^\nabla(X) := u^{-1} \frac{\nabla u(s)}{ds} \Big|_{s=0}$$

where $u(\cdot)$ is a differentiable curve on $\mathcal{O}(M)$ such that $u(0) = u$ and $\frac{du(s)}{ds} \Big|_{s=0} = X$. For any $\xi \in \mathbb{R}^d$, $\frac{\nabla u(s)}{ds} \Big|_{s=0} \xi := \frac{\nabla u(s)\xi}{ds} \Big|_{s=0}$ is the covariant derivative of $u(\cdot)\xi$ along $\pi(u(\cdot))$ at $\pi(u)$.

Definition 2.3 (Horizontal Bundle \mathcal{H}) *Given a connection form ω^∇ , the horizontal bundle $\mathcal{H} \subset T\mathcal{O}(M)$ is defined to be the kernel of ω^∇ .*

Definition 2.4 *For any $a \in \mathbb{R}^d$, define the horizontal lift $B_a \in \Gamma(\mathcal{H})$ in the following way: for any $u \in \mathcal{O}(M)$, $B_a(u) \in \mathcal{H}_u \subset T_u \mathcal{O}(M)$ is uniquely determined by*

$$\omega_u^\nabla(B_a(u)) = 0 \text{ and } \pi_*(B_a(u)) = ua.$$

Definition 2.5 (Horizontal Lift of a Path) *For any $\sigma \in H(M)$, a curve $u : [0, 1] \rightarrow \mathcal{O}(M)$ is said to be a horizontal lift of σ if $\pi \circ u = \sigma$ and $u'(s) \in \mathcal{H}_{u(s)} \forall s \in [0, 1]$.*

Remark 2.6 *In this paper we only consider horizontal lift with fixed start point $u_0 \in \pi^{-1}(\sigma(0))$. Under this assumption, given $\sigma \in H(M)$, its horizontal lift $u(\sigma, \cdot)$ is unique.*

We denote u by $\psi(\sigma)$ and call ψ the **horizontal lift map**.

Definition 2.7 (Development Map) Given $w \in H(\mathbb{R}^d)$, the solution to the ordinary differential equation

$$du(s) = \sum_{i=1}^d B_{e_i}(u(s)) dw^i(s), u(0) = u_0$$

is defined to be the **development** of w and we will denote this map $w \rightarrow u$ by η , i.e. $\eta(w) = u$. Here $\{e_i\}_{i=1}^d$ is the standard basis of \mathbb{R}^d .

Definition 2.8 (Rolling Map) $\phi = \pi \circ \eta : H(\mathbb{R}^d) \rightarrow H(M)$ is said to be the rolling map to $H(M)$.

Definition 2.9 (Anti-rolling Map) Given $\sigma \in H(M)$ with $u = \psi(\sigma)$. The anti-rolling of σ is a curve $w \in H(\mathbb{R}^d)$ defined by:

$$w_t = \int_0^t u_s^{-1} \sigma'_s ds$$

Remark 2.10 It is not hard to see $w = \phi^{-1}(\sigma)$ and $u(\sigma, s)u_0^{-1}$ is the parallel translation along $\sigma \in H(M)$.

A stochastic version of the maps defined above is needed to specify the differential structure on $(W_o(M), \nu)$. It also provides tools that allow the transition between classical Wiener space and curved Wiener space. Since the development maps on the smooth category are defined through ordinary differential equations, a natural way to introduce probability is to replace ODEs by (Stratonovich) stochastic differential equations.

First we set up some measure theoretic notations and conventions. Suppose $(\Omega, \{\mathcal{G}_s\}, \mathcal{G}, P)$ is a filtered measurable space with a finite measure P . For any \mathcal{G} -measurable function f , we use $P(f)$ and $\mathbb{E}_P[f]$ (if P is a probability measure) to denote the integral $\int_{\Omega} f dP$. Given two filtered measurable spaces $(\Omega, \{\mathcal{G}_s\}, \mathcal{G}, P)$ and $(\Omega', \{\mathcal{G}'_s\}, \mathcal{G}', P')$ and a \mathcal{G}/\mathcal{G}' measurable map $f : \Omega \rightarrow \Omega'$, the law of f under P is the push-forward measure $f_*P(\cdot) := P(f^{-1}(\cdot))$. We will be mostly interested in the path spaces $W_o(M)$, $W_0(\mathbb{R}^d)$ and $W_{u_0}(\mathcal{O}(M))$.

Definition 2.11 Given a Riemannian manifold Y , for any $s \in [0, 1]$ let $\Sigma_s : W_y(Y) \rightarrow Y$ be the **coordinate functions** given by $\Sigma_s(\sigma) = \sigma(s)$.

We will often view Σ as a map from $W_y(Y)$ to $W_y(Y)$ in the following way: for any $\sigma \in W_y(Y)$ and $s \in [0, 1]$, $\Sigma(\sigma)(s) = \Sigma_s(\sigma)$. Let \mathcal{F}_s^o be the σ -algebra generated by $\{\Sigma_\tau : \tau \leq s\}$. We use \mathcal{F}_1^o as the raw σ -algebra and $\{\mathcal{F}_s^o\}_{0 \leq s \leq 1}$ as the filtration on $W_y(Y)$. The next theorem defines the Wiener measure ν on $(W_y(Y), \mathcal{F}_1^o)$.

Theorem 2.12 (Wiener measure) Assume Y is a stochastically complete Riemannian manifold, then there exists a unique probability measure ν on $(W_y(Y), \mathcal{F}_1^o)$ which is uniquely determined by its finite dimensional distributions as follows. For any partition $0 = s_0 < s_1 < \dots < s_{n-1} < s_n = 1$ of $[0, 1]$ and bounded functions $f : Y^n \rightarrow \mathbb{R}$;

$$\nu(f(\Sigma_{s_1}, \dots, \Sigma_{s_n})) = \int_{Y^n} f(x_1, \dots, x_n) \prod_{i=1}^n p_{\Delta s_i}(x_{i-1}, x_i) dx_1 \dots dx_n \quad (2.1)$$

where $p_t(\cdot, \cdot)$ is the heat kernel on Y associated with $\frac{1}{2}\Delta_g$, $\Delta_i = s_i - s_{i-1}$ for $1 \leq i \leq n$.

Definition 2.13 (Brownian motion) A stochastic process $X : (\Omega, \mathcal{G}_s, \{\mathcal{G}\}, P) \rightarrow (W_y(Y), \nu)$ is said to be a Brownian motion on Y if the law of X is ν i.e. $X_*P := P \circ X^{-1} = \nu$.

Remark 2.14 From Theorem 2.12 it is clear that the law of the adapted process $\Sigma : W_y(Y) \rightarrow W_y(Y)$ is ν and Σ is a Brownian motion. We will call Σ the **canonical Brownian motion** on Y .

Remark 2.15 Using Theorem 2.12, we can construct Wiener measure on $W_0(\mathbb{R}^d)$, $W_o(M)$ and $W_{u_0}(\mathcal{O}(M))$ respectively. In order to avoid ambiguity from moving between $W_0(\mathbb{R}^d)$ and $W_o(M)$, as is mentioned at the beginning of the introduction, we fix the symbol μ as the Wiener measure on $W_0(\mathbb{R}^d)$ and reserve the symbol ν as the Wiener measure on $W_o(M)$. Meanwhile we reserve Σ as the canonical Brownian motion on M .

Theorem 2.16 (Stochastic Horizontal Lift of Brownian Motion) If Σ is the canonical Brownian motion on M , then there exists a unique (up to ν -equivalence) $\tilde{u} \in W_{u_0}(\mathcal{O}(M))$ such that

$$\pi(\tilde{u}_s) = \Sigma_s. \quad (2.2)$$

Proof. See Theorem 2.3.5 in [17] ■

Definition 2.17 (Stochastic Anti-rolling Map) If Σ is the canonical Brownian motion on M , then the stochastic anti-rolling β of Σ is defined by,

$$d\beta_s = \tilde{u}_s^{-1} \delta \Sigma_s, \beta_0 = 0. \quad (2.3)$$

\tilde{u} and β defined above are linked through the (stochastic) development map.

Definition 2.18 (Stochastic Development Map) Let \tilde{u} and β be as defined in Theorem 2.16 and Definition 2.17, then \tilde{u} satisfies the following SDE driven by β ,

$$d\tilde{u}_s = \sum_{i=1}^d B_{e_i}(\tilde{u}_s) \delta \beta_s, \tilde{u}(0) = u_0,$$

and \tilde{u} is said to be the development of β .

Fact 2.19 The following facts are well known, the proofs may be found in the references listed at the beginning of this section, for example, Theorem 3.3 in [8].

- ϕ is a diffeomorphism from $H(\mathbb{R}^d)$ to $H(M)$,
- β is a Brownian motion on $(W_o(\mathbb{R}^d), \mu)$.

From now on some notations are fixed for the convenience of consistency.

Notation 2.20 For any $\sigma \in H(M)$, $u_{(\cdot)}(\sigma) \in H_{u_0}(\mathcal{O}(M))$ is its horizontal lift and $b_{(\cdot)}(\sigma) \in H(\mathbb{R}^d)$ is its anti-rolling. Recall that $\{\Sigma_s\}_{0 \leq s \leq 1}$ is fixed to be the canonical Brownian motion on $(W_o(M), \nu)$. We also fix $\beta(\cdot)$ to be the stochastic anti-rolling of Σ , (which is a Brownian motion on \mathbb{R}^d) and $\tilde{u}(\cdot)$ to be the stochastic horizontal lift of Σ .

Notation 2.21 (Geometric Notation)

- **curvature tensor** For any $X, Y, Z \in \Gamma(TM)$, define the (Riemann) curvature tensor $R : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(\text{End}(TM))$ to be:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

- For any $\sigma \in H(M)$, define $R_{u(\sigma, s)}(\cdot, \cdot) \cdot$ to be a map from $\mathbb{R}^d \otimes \mathbb{R}^d$ to $\text{End}(\mathbb{R}^d)$ given by;

$$R_{u(\sigma, s)}(a, b) \cdot = u(\sigma, s)^{-1} R(u(\sigma, s)a, u(\sigma, s)b) u(\sigma, s) \quad \forall a, b \in \mathbb{R}^d. \quad (2.4)$$

where R is the curvature tensor of M . Similarly we define $R_{\tilde{u}(\sigma, s)}(\cdot, \cdot) \cdot$ to be a random map (up to ν -equivalence) from $\mathbb{R}^d \otimes \mathbb{R}^d$ to \mathbb{R}^d as follows:

$$R_{\tilde{u}(\sigma, s)}(\cdot, \cdot) \cdot = \tilde{u}(\sigma, s)^{-1} R(\tilde{u}(\sigma, s) \cdot, \tilde{u}(\sigma, s) \cdot) \tilde{u}(\sigma, s). \quad (2.5)$$

- $Ric(\cdot) := \sum_{i=1}^d R(v_i, \cdot) v_i$ is the Ricci curvature tensor on M . Here $\{v_i\}_{i=1}^d$ is an orthonormal basis of proper tangent space. Using $u(\sigma, s)$ or $\tilde{u}(\sigma, s)$ to pull back R as in (2.4) and (2.5), we can define $Ric_{u(\sigma, s)}$ and $Ric_{\tilde{u}(\sigma, s)}$ to be maps (random maps) from \mathbb{R}^d to \mathbb{R}^d .

Convention 2.22 Since most of our results require a curvature bound, it would be convenient to fix a symbol N for it, i.e. $\|R\| \leq N$ when it is viewed as a tensor of order 4. Following this manner, we have $\|Ric\| \leq (d-1)N$. A generic constant will be denoted by C , it can vary from line to line. Sometimes $C_{(\cdot)}$ or $C(\cdot)$ are used to specify its dependence on some parameters.

Definition 2.23 $f : W_o(M) \mapsto \mathbb{R}$ is a **cylinder function** if there exists a partition

$$\mathcal{P} := \{0 < s_1 < \dots < s_n \leq 1\}$$

of $[0, 1]$ and a function $F : C^m(M^n, \mathbb{R})$ such that

$$f = F(\Sigma_{s_1}, \Sigma_{s_2}, \dots, \Sigma_{s_n}).$$

We denote this space by \mathcal{FC}^m .

Notation 2.24 Denote

$$\mathcal{FC}_b^1 := \{f := F(\Sigma) \in \mathcal{FC}^1, F \text{ and all its partial differentials } \text{grad}_i F \text{ are bounded}\}.$$

Notation 2.25 Given a measurable function $h : H(M) \rightarrow H(\mathbb{R}^d)$, denote

$$X^h(\sigma, s) := u(\sigma, s) h(\sigma, s).$$

With this notation, we can express, for any $\sigma \in H(M)$,

$$T_\sigma H(M) = \{X^h \mid h : H(M) \rightarrow H(\mathbb{R}^d) \text{ is measurable}\}.$$

3 The Orthogonal Lift \tilde{X} of X on $H(M)$ and Its Stochastic Extension

3.1 Damped Metrics and Adjoints

Notation 3.1 For any $r, s \in \mathbb{N}$, the (r, s) -tensor bundle on M is denoted by $T^{r,s}M$.

Given $\Lambda \in \Gamma(T^{1,1}M)$, we can define a damped metric on $H(M)$ by replacing Ric with Λ in Definition 1.8. Furthermore, for any $\sigma \in H(M)$, using parallel translation $u(\sigma, \cdot)$, one can obtain an isometry between $(T_\sigma H(M), \langle \cdot, \cdot \rangle_\Lambda)$ and $(H(\mathbb{R}^d), \langle \cdot, \cdot \rangle_\alpha)$, where $\alpha(\cdot) = u(\cdot)^{-1} \circ \Lambda \circ u(\cdot) \in C([0, 1], \text{End}(\mathbb{R}^d))$. So in order to prove Theorem 1.11, there is no more difficulty in considering the following more general metric on $H(\mathbb{R}^d)$.

Definition 3.2 (α -inner product) Let $\alpha(t) \in \text{End}(\mathbb{R}^d)$ be a continuously varying matrix valued function. For $h, k \in H(\mathbb{R}^d)$ let

$$\langle h, k \rangle_\alpha := \int_0^1 \left(\frac{d}{dt} h(t) + \alpha(t) h(t) \right) \cdot \left(\frac{d}{dt} k(t) + \alpha(t) k(t) \right) dt.$$

Remark 3.3 We denote the norm induced by α -inner product by $\|\cdot\|_\alpha$, differentiating from the notation $\|\cdot\|_{H(\mathbb{R}^d)}$ for the norm induced by the H^1 -inner product: $\langle h, k \rangle_{H^1} = \int_0^1 h'(s) \cdot k'(s) ds$.

For the moment, let $E_1 : H(\mathbb{R}^d) \rightarrow \mathbb{R}^d$ be the end point evaluation map in the case where $M = \mathbb{R}^d$. Let $E_1^* : \mathbb{R}^d \rightarrow H(\mathbb{R}^d)$ be the adjoint of E_1 with respect to the α -inner product, i.e. for any $a \in \mathbb{R}^d$ and $h \in H(\mathbb{R}^d)$,

$$\langle E_1 h, a \rangle_{\mathbb{R}^d} = \langle h, (E_1^*) a \rangle_\alpha.$$

The next theorem computes E_1^* which is crucial in constructing the orthogonal lift in Subsection 3.2.

Theorem 3.4 Let $a \in \mathbb{R}^d$ and $\alpha(t)$ be as in Definition 3.2, then $E_1^* a \in H(\mathbb{R}^d)$ is given by

$$(E_1^* a)(t) = \left(S(t) \int_0^t [S(s)^* S(s)]^{-1} S(1)^* ds \right) a. \quad (3.1)$$

where $S(t) \in \text{Aut}(\mathbb{R}^d)$ solves

$$\frac{d}{dt} S(t) + \alpha(t) S(t) = 0 \text{ with } S(0) = I,$$

$S(t)^*$ is the conjugate transpose of $S(t)$.

Proof. Notice that if $h(t) = S(t) w(t)$ with $w(\cdot) \in H(\mathbb{R}^d)$, then

$$\left(\frac{d}{dt} + \alpha(t) \right) h(t) = \left(\frac{d}{dt} + \alpha(t) \right) [S(t) w(t)] = \left[\left(\frac{d}{dt} + \alpha(t) \right) S(t) \right] w(t) + S(t) \dot{w}(t) = S(t) \dot{w}(t).$$

And in particular,

$$\langle Sv, Sw \rangle_\alpha = \int_0^1 S(t) \dot{v}(t) \cdot S(t) \dot{w}(t) dt.$$

Using Lemma A.1 we know $S(t) \in \text{Aut}(\mathbb{R}^d)$. Given $a \in \mathbb{R}^d$, let $w(t) = E_1^* a$ and define $v(t) := S(t)^{-1} w(t)$ so that $E_1^* a = S(t) v(t)$. Then by the definition of the adjoint we find,

$$\begin{aligned} \int_0^1 S(t) \dot{v}(t) \cdot S(t) \dot{w}(t) dt &= \langle Sv, Sw \rangle_\alpha = \langle E_1^* a, Sw \rangle_\alpha = a \cdot E_1(Sw) \\ &= a \cdot S(1) w(1) = \int_0^1 S(1)^* a \cdot \dot{w}(t) dt \end{aligned}$$

As $w \in H(\mathbb{R}^d)$ is arbitrary we may conclude that

$$S(t)^* S(t) \dot{v}(t) = S(1)^* a \implies v(t) = \int_0^t [S(s)^* S(s)]^{-1} S(1)^* a ds$$

which proves (3.1). ■

Theorem 3.5 If $a \in \mathbb{R}^d$, then $h(\cdot) \in H(\mathbb{R}^d)$ defined by

$$h(t) := S(t) \left(\int_0^t [S(s)^* S(s)]^{-1} ds \right) \left(\int_0^1 [S(s)^* S(s)]^{-1} ds \right)^{-1} S(1)^{-1} a, \quad (3.2)$$

is the minimal length element of $H(\mathbb{R}^d)$ such that $E_1 h = a$, i.e.

$$\|h\|_\alpha = \inf \{ \|k\|_\alpha \mid k(\cdot) \in H(\mathbb{R}^d), E_1 k = a \}.$$

Proof. Since $H(\mathbb{R}^d) = \text{Nul}(E_1)^\perp \oplus \text{Nul}(E_1)$, we have $E_1 h = a \implies E_1 h_k = a$ and $\|h\|_\alpha \geq \|h_k\|_\alpha$ where h_k is the orthogonal projection of h onto $\text{Nul}(E_1)^\perp$. So we are looking for the element, $h \in H(\mathbb{R}^d)$, such that $E_1 h = a$ and $h \in \text{Nul}(E_1)^\perp = \text{Ran}(E_1^*)$. In other words we should have $h = E_1^* v$ for some $v \in \mathbb{R}^d$. Thus, using Eq.(3.1), we need to demand that

$$a = E_1 E_1^* v = (E_1^* v)(1) = \left(S(1) \int_0^1 [S(s)^* S(s)]^{-1} S(1)^* ds \right) v,$$

i.e.

$$v = \left(S(1) \int_0^1 [S(s)^* S(s)]^{-1} S(1)^* ds \right)^{-1} a.$$

Here we have used Lemma A.1 to show $S(1)$ and $\int_0^1 [S(s)^* S(s)]^{-1} ds$ are invertible.

It then follows that

$$\begin{aligned} h(t) &= E_1^* \left(S(1) \int_0^1 [S(s)^* S(s)]^{-1} S(1)^* ds \right)^{-1} a \\ &= \left(S(t) \int_0^t [S(s)^* S(s)]^{-1} S(1)^* ds \right) \left(S(1) \int_0^1 [S(s)^* S(s)]^{-1} S(1)^* ds \right)^{-1} a \end{aligned}$$

which is equivalent to Eq.(3.2). ■

Remark 3.6 The expression in (3.2) matches the well known result for damped metric where $\alpha = \frac{1}{2} \text{Ric}_u$. Further observe that if $\alpha(t) = 0$ (i.e. we are in the flat case) then $S(t) = I$ and the above expression reduces to $h(t) = ta$ as we know to be the correct result.

3.2 The Orthogonal Lift \tilde{X} on $H(M)$

In this subsection we construct the orthogonal lift $\tilde{X} \in \Gamma(TH(M))$ of $X \in \Gamma(TM)$ which is defined to be the minimal length element in $\Gamma(TH(M))$ relative to the damped metric introduced in Definition 1.8.

Definition 3.7 For each $\sigma \in H(M)$, recall that $u_s(\sigma)$ is the horizontal lift of σ . Denote by $T_{(\cdot)} : H(M) \rightarrow \text{End}(\mathbb{R}^d)$ the solution to the following initial value problem:

$$\begin{cases} \frac{d}{ds} T_s + \frac{1}{2} \text{Ric}_{u_s} T_s = 0 \\ T_0 = I. \end{cases} \quad (3.3)$$

Lemma 3.8 For all $s \in [0, 1]$, T_s is invertible. Further both $\sup_{0 \leq s \leq 1} \|T_s\|$ and $\sup_{0 \leq s \leq 1} \|T_s^{-1}\|$ are bounded by $e^{\frac{1}{2}(d-1)N}$, where $(d-1)N$ is a bound of $\|\text{Ric}\|$.

Proof. Apply Lemma A.1 with $\alpha(s) = -\frac{1}{2} \text{Ric}_{u_s}$, one get T_s is invertible $\forall s \in [0, 1]$ and T_s^{-1} satisfies the following ODE,

$$\begin{cases} \frac{d}{ds} U_s = \frac{1}{2} U_s \text{Ric}_{u_s} \\ U_0 = I. \end{cases} \quad (3.4)$$

The stated bounds now follow by Gronwall's inequality and the boundedness of curvature tensor. ■

Definition 3.9 Let $\mathbf{K} : [0, 1] \times H(M) \rightarrow \text{End}(\mathbb{R}^d)$ be defined by

$$\mathbf{K}_s := T_s \left[\int_0^s T_r^{-1} (T_r^{-1})^* dr \right] T_1^*. \quad (3.5)$$

Lemma 3.10 \mathbf{K}_1 is invertible and $\|\mathbf{K}_1^{-1}\| \leq e^{(d-1)N}$, provided $\|\text{Ric}\| \leq (d-1)N$.

Proof. Since

$$\mathbf{K}_1 := \int_0^1 (T_1 T_r^{-1}) (T_1 T_r^{-1})^* dr$$

is a symmetric positive semi-definite operator such that

$$\langle \mathbf{K}_1 v, v \rangle = \int_0^1 \left\| (T_1 T_r^{-1})^* v \right\|^2 dr \quad \forall v \in \mathbb{C}^d.$$

Apply Lemma 3.8 to the expression given;

$$\langle \mathbf{K}_1 v, v \rangle \geq \int_0^1 e^{-(d-1)N} \left\| (T_r^{-1})^* v \right\|^2 dr \geq \int_0^1 e^{-2(d-1)N} \|v\|^2 dr = e^{-2(d-1)N} \|v\|^2$$

from which it follows that $\text{eig}(\mathbf{K}_1) \subset [e^{-(d-1)N}, \infty)$ and $\|\mathbf{K}_1^{-1}\| = \frac{1}{\min\{\lambda : \lambda \in \text{eig}(\mathbf{K}_1)\}} \leq e^{(d-1)N}$. ■

Definition 3.11 Let $X \in \Gamma(TM)$, define two maps $H : H(M) \rightarrow \mathbb{R}^d$ and $J : [0, 1] \times H(M) \rightarrow \mathbb{R}^d$ as follows,

$$H(\sigma) = u_1^{-1}(\sigma) X \circ E_1(\sigma) \quad (3.6)$$

and

$$J(\sigma, s) := J_s(\sigma) := \mathbf{K}_s(\sigma) \mathbf{K}_1^{-1}(\sigma) H(\sigma). \quad (3.7)$$

Theorem 3.12 Given $X \in \Gamma(TM)$, the minimal length lift \tilde{X} relative to the damped metric in Definition 1.8 of X to $\Gamma(TH(M))$, is given by $\tilde{X} = X^J$. Further we know that J_s is the solution to the following ODE:

$$J'_s = -\frac{1}{2} Ric_{u_s} J_s + \phi_s, \quad J_0 = 0$$

where $\phi_s = (T_1 T_s^{-1})^* \mathbf{K}_1^{-1} H = (T_s^{-1})^* \left[\int_0^1 T_r^{-1} (T_r^{-1})^* dr \right]^{-1} T_1^{-1} H$.

Proof. Apply Theorem 3.5 with $\alpha_s = \frac{1}{2} Ric_{u_s}$. ■

The following construction gives rise to a stochastic extension of \tilde{X} to a Cameron-Martin vector field on $W_o(M)$. The definition of Cameron-Martin vector field is given right below. Its properties are further explored in the next section.

Recall from Notation 2.20 that \tilde{u} is the stochastic horizontal lift of the canonical Brownian motion Σ on M . Mimicking the tangent bundle $TH(M)$ of $H(M)$ as expressed in Notation 2.25, we define a Cameron-Martin vector field (not necessarily adapted) as follows.

Definition 3.13 A **Cameron-Martin process**, h , is an \mathbb{R}^d -valued process on $W_o(M)$ such that $s \rightarrow h(s)$ is in $H(\mathbb{R}^d)$ ν -a.s. A TM -valued process Y on $(W_o(M), \nu)$ is called a **Cameron-Martin vector field** (denote this space by \mathcal{X}) if $\pi(Y_s) = \Sigma_s$ ν -a.s. and there exists a Cameron-Martin process $h(\cdot)$ such that $Y(s) = \tilde{u}_s h_s \forall s \in [0, 1]$ ν -a.s. with

$$\langle Y, Y \rangle_{\mathcal{X}} := \mathbb{E} \left[\|h\|_{H(\mathbb{R}^d)}^2 \right] < \infty.$$

We will write $X^h = Y$ to highlight this representation and X^h is called adapted if h is adapted.

Definition 3.14 Define $\tilde{T}_{(\cdot)} : [0, 1] \times W_o(M) \rightarrow End(\mathbb{R}^d)$ to be the solution to the following initial value problem:

$$\begin{cases} \frac{d}{ds} \tilde{T}_s + \frac{1}{2} Ric_{\tilde{u}_s} \tilde{T}_s = 0 \\ \tilde{T}_0 = I \end{cases} \quad (3.8)$$

Definition 3.15 Using \tilde{T}_s , we define $\tilde{\mathbf{K}} : [0, 1] \times W_o(M) \rightarrow End(\mathbb{R}^d)$:

$$\tilde{\mathbf{K}}_s := \tilde{T}_s \left[\int_0^s \tilde{T}_r^{-1} (\tilde{T}_r^{-1})^* dr \right] \tilde{T}_1^*. \quad (3.9)$$

Remark 3.16 Following the same arguments used in Lemma 3.8 and 3.10, one can see the bounds obtained there still hold for \tilde{T} and $\tilde{\mathbf{K}}$ ν -a.s.

Definition 3.17 For each $X \in \Gamma(TM)$ define two maps $\tilde{H} : W_o(M) \rightarrow \mathbb{R}^d$ and $\tilde{J} : W_o(M) \rightarrow H(\mathbb{R}^d)$ by

$$\tilde{H} = \tilde{u}_1^{-1} X \circ E_1 \quad (3.10)$$

and

$$\tilde{J}_s := \tilde{\mathbf{K}}_s \tilde{\mathbf{K}}_1^{-1} \tilde{H} \text{ for } s \in [0, 1]. \quad (3.11)$$

Notation 3.18 Given a measurable function $h : W_o(M) \rightarrow H(\mathbb{R}^d)$, let $Z_h : W_o(M) \rightarrow H(\mathbb{R}^d)$ be the solution to the following initial value problem:

$$\begin{cases} Z_h'(s) = -\frac{1}{2} Ric_{\tilde{u}_s} Z_h(s) + h'_s \\ Z_h(0) = 0. \end{cases}$$

Definition 3.19 (Orthogonal Lift on $W_o(M)$) For any $X \in \Gamma(TM)$, define $\tilde{X} \in \mathcal{X}$ as follows,

$$\tilde{X}_s = X_s^{Z_\Phi} := \tilde{u}_s Z_\Phi(s) \text{ for } 0 \leq s \leq 1$$

where

$$\Phi_s = \int_0^s \left(\tilde{T}_\tau^{-1} \right)^* \left[\int_0^1 \left(\tilde{T}_r^* \tilde{T}_r \right)^{-1} dr \right]^{-1} \tilde{T}_1^{-1} \tilde{H} d\tau.$$

In the next section we will specify how this Cameron-Martin vector field act on geometric Wiener functionals.

4 A Differential Calculus on $W_o(M)$ for \tilde{X}

4.1 Review of Calculus on Wiener Space

First we review some classical results for adapted Cameron-Martin vector field where (approximate) flows can be constructed.

Definition 4.1 (Vector Valued Brownian Semimartingale) Let V be a finite dimensional vector space. A function $f : W_o(M) \times [0, 1] \rightarrow V$ is called a Brownian semimartingale if f has the following representation:

$$f(s) = \int_0^s Q_\tau d\beta_\tau + \int_0^s r_\tau d\tau$$

where (Q_s, r_s) is a predictable process with values in $\text{Hom}(\mathbb{R}^d, V) \times V$. We will call (Q_s, r_s) the kernels of f .

Definition 4.2 (R^q and \mathcal{H}^q Space) For each $q \in [1, \infty]$, $f : W_o(M) \times [0, 1] \rightarrow V$ jointly measurable, we define the root mean square norm in $L^q(W_o(M), \nu)$ to be:

$$\|f\|_{R^q(V)} \equiv \left\| \left(\int_0^1 |f(\cdot, s)|_V^2 ds \right)^{\frac{1}{2}} \right\|_{L^q(W_o(M), \nu)}.$$

Let R^q be the space of all $f : W_o(M) \times [0, 1] \rightarrow V$ such that $\|f\|_{R^q} < \infty$ and let \mathcal{H}^q be the space of all Brownian semimartingales such that

$$\|f\|_{\mathcal{H}^q} := \|Q^f\|_{R^q} + \|r^f\|_{R^q} < \infty.$$

Here we suppress the range space V as it should be easily determined by the context.

Definition 4.3 (S^q and \mathcal{B}^q Space) For each $q \in [1, \infty]$, $f : W_o(M) \times [0, 1] \rightarrow V$ jointly measurable, we define the supremum norm in $L^q(W_o(M), \nu)$ to be:

$$\|f\|_{S^q(V)} \equiv \|f^*\|_{L^q(W_o(M), \nu)}$$

where f^* is the essential supremum of $s \rightarrow f(\cdot, s)$ relative to Lebesgue measure on $[0, 1]$. Let S^q be the space of all $f : W_o(M) \times [0, 1] \rightarrow V$ such that $s \rightarrow f(s, \cdot) : [0, 1] \rightarrow V$ is continuous ν -a.s. and $\|f\|_{S^q} < \infty$ and let \mathcal{B}^q be the space of all Brownian semimartingales such that

$$\|f\|_{\mathcal{B}^q} := \|Q^f\|_{S^q} + \|r^f\|_{S^q} < \infty.$$

Lemma 4.4 For any $q \in [1, \infty)$, $f : W_o(M) \times [0, 1] \rightarrow V$ such that the following norms make sense, we have

- $\|f\|_{R^q(V)} \leq \|f\|_{S^q(V)},$
- $\|f\|_{\mathcal{H}^q(V)} \leq \|f\|_{\mathcal{B}^q(V)},$

- $\|f\|_{S^q(V)} \leq C_q \|f\|_{\mathcal{H}^q(V)}$ for some constant $C_q > 0$.

Proof. The first two items are trivial, so we will only prove the last item.

Since f has the following representation

$$f_s = \int_0^s Q_\tau d\beta_\tau + \int_0^s r_\tau d\tau,$$

for any $q \in [1, \infty)$, we have

$$|f_s|^q \leq C_q \left(\left| \int_0^s Q_\tau d\beta_\tau \right|^q + \left(\int_0^s |r_\tau| d\tau \right)^q \right)$$

and thus

$$|f^*|^q \leq C_q \left(\sup_{0 \leq s \leq 1} \left| \int_0^s Q_\tau d\beta_\tau \right|^q + \left(\int_0^1 |r_\tau|^2 d\tau \right)^{\frac{q}{2}} \right). \quad (4.1)$$

From Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}_\nu \left[\sup_{0 \leq s \leq 1} \left| \int_0^s Q_\tau d\beta_\tau \right|^q \right] \leq C_q \|Q\|_{R^q}^q,$$

then taking expectations on Eq.(4.1) we have

$$\|f\|_{S^q} \leq C_q (\|Q\|_{R^q} + \|r\|_{R^q}) = C_q \|f\|_{\mathcal{H}^q}.$$

■

Definition 4.5 (Adapted Vector Field) An adapted vector field on $W_o(M)$ is an \mathbb{R}^d -valued Brownian semimartingale with predictable kernels $Q \in \mathfrak{so}(d)$ and $r \in L^2[0, 1]$ ν -a.s. We denote the space of **adapted vector fields** by \mathcal{V} and let \mathcal{V}^q be $\mathcal{V} \cap \mathcal{H}^q$, $q \in [1, \infty]$.

Notation 4.6 We will use the following notations in this paper: $S^{\infty-} := \cap_{q \geq 1} S^q$, $\mathcal{H}^{\infty-} := \cap_{q \geq 1} \mathcal{H}^q$, $\mathcal{B}^{\infty-} = \cap_{q \geq 1} \mathcal{B}^q$ and $\mathcal{V}^{\infty-} = \mathcal{V} \cap \mathcal{H}^{\infty-}$.

Theorem 4.7 (Approximate Flow) Let X^h be a Cameron-Martin vector field with $h \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$, $t \in \mathbb{R}$, then there exists a map $E(tX^h) : W_o(M) \rightarrow W_o(M)$ such that the law of $E(tX^h)$ is equivalent to ν and

$$\frac{d}{dt} \big|_0 E(tX^h) = X^h \text{ in } \mathcal{B}^{\infty-}.$$

Proof. See Corollary 4.6 in [7]. ■

Using the approximate flow, we will specify a domain of an adapted Cameron-Martin vector field with the aim of setting up an integration by parts formula. A remark about other possible domains are provided after the definition below.

Definition 4.8 Let X^h be an adapted Cameron-Martin vector field with $h \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$ and let $E(tX^h) : W_o(M) \rightarrow W_o(M)$ be its approximate flow, then we define the domain of X^h to be

$$\mathcal{D}(X^h) := \left\{ f \in L^{\infty-}(W_o(M), \nu), \frac{d}{dt} \big|_0 f(E(tX^h)) \text{ exists in } L^{\infty-}(W_o(M), \nu) \right\} \subset L^2(W_o(M))$$

and define $X^h f := \frac{d}{dt} \big|_0 f(E(tX^h))$.

Remark 4.9 (H-derivative) The notion of differentiability in Definition 4.8 is weaker than the one defined using H-derivative which allows a Sobolev type analysis on $(W_o(M), \nu)$. However this definition is sufficient to admit an integration by parts formula, see Lemma 4.23. Here we provide a very rough picture of how the H-derivative is defined.

Given $f \in \mathcal{FC}_b^1$, define the **gradient operator** $Df \in \mathcal{X}$ as follows,

$$D_s f := \tilde{u}_s \sum_{i=1}^n (s \wedge s_i) \tilde{u}_{s_i}^{-1} \text{grad}_i F \quad (4.2)$$

where $F(\Sigma_{s_1}, \dots, \Sigma_{s_n})$ is a representation of f and $\text{grad}_i F$ is the differential of F with respect to the i th variable.

Since \mathcal{FC}_b^1 is dense in $L^q(W_o(M), \nu) \forall q \geq 1$, we know D is a densely defined operator from $L^q(W_o(M), \nu)$ to \mathcal{X} . Furthermore, it is well-known that D is closable in $L^q(W_o(M), \nu) \forall q \geq 1$ and the domain of its extension is a Sobolev space of index $(1, q)$ on $W_o(M)$. (We will denote this space by $W_1^q(M)$.) If we treat D as an operator from $L^{\infty-}(W_o(M)) := \cap_{q \geq 1} L^q(W_o(M))$ to \mathcal{X} with domain $\mathcal{D}(D) := W_1^{\infty-}(M) := \cap_{q \geq 1} W_1^q(M)$, then for any $X \in \mathcal{X}$, we may define $Xf := \langle Df, X \rangle_{G^1}$ and require its domain $\mathcal{D}(X)$ to be $W_1^{\infty-}(M)$. However if X is not adapted, it is not known if X is in the domain of $D^* : \mathcal{X} \rightarrow W_1^{\infty-}(M)$ — a fact that easily gives rise to integration by parts. There is also the issue of dependence on initial domain when taking closure for H -derivative on curved Wiener space.

The following example shows some advantages of Definition 4.8: basically one can show that a class of so called generalized cylinder functions are X^h differentiable by explicit computations. This content is summarized from [7].

Definition 4.10 $f : W_o(M) \mapsto \mathbb{R}$ is called a **generalized cylinder function** if there exists a partition

$$\mathcal{P} := \{0 < s_1 < \dots < s_n \leq 1\}$$

of $[0, 1]$ and a bounded function $F \in C^m(\mathcal{O}(M)^n, \mathbb{R})$ such that:

$$f = F(\tilde{u}_{s_1}, \tilde{u}_{s_2}, \dots, \tilde{u}_{s_n}) \quad \nu - a.s.$$

We further require all the partial differentials of F to be bounded and denote this space by \mathcal{GFC}^m .

Notation 4.11 Given $k : W_o(M) \rightarrow H(\mathbb{R}^d)$, denote $\int_0^s R_{\tilde{u}_r}(k_r, \delta\beta_r)$ by $A_s\langle k \rangle$ when the integral makes sense, here δ is the stratonovich differential.

Notation 4.12 Suppose $F \in C(\mathcal{O}(M)^n)$ and $\mathcal{P} = \{0 < s_1 < \dots < s_n \leq 1\}$ is a partition of $[0, 1]$, set

$$F(u) = F(u_{s_1}, \dots, u_{s_n}),$$

then for $A : [0, 1] \rightarrow \mathfrak{so}(d)$ and $h : [0, 1] \rightarrow \mathbb{R}^d$, set

$$F'(u) \langle A + h \rangle := \frac{d}{dt} \big|_0 F(ue^{tA}) + \frac{d}{dt} \big|_0 F(e^{tB_h}(u))$$

where $ue^{tA}(s) = u_s e^{tA_s} \in \mathcal{O}(M)$ and $e^{tB_h}(u)(s) = e^{tB_{h,s}}(u_s) \in \mathcal{O}(M)$.

Theorem 4.13 If $h \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$, then $\mathcal{GFC}^1 \subset \mathcal{D}(X^h)$. In more detail, if $f = F(\tilde{u}) \in \mathcal{GFC}^1$, then

$$X^h f = F'(\tilde{u}) \langle -A \langle h \rangle + h \rangle \quad \nu - a.s. \quad (4.3)$$

Moreover, if $g \in \mathcal{D}(X^h)$, then

$$\mathbb{E}_\nu [X^h f \cdot g] = \mathbb{E}_\nu [f \cdot (X^h)^{tr, \nu} g] \quad (4.4)$$

where $(X^h)^{tr, \nu} := -X^h + \int_0^1 \langle h'_s, d\beta_s \rangle$.

Proof. See Proposition 4.10 in [7]. ■

We now construct a class of Cameron-Martin vector field and use it as a basis to expand the orthogonal lift \tilde{X} defined in Definition 3.19.

Notation 4.14 Recall from Notation 3.18 that Z_h satisfies the following ODE,

$$Z'_h(s) = -\frac{1}{2}Ric_{\tilde{u}_s}Z_h(s) + h'_s \text{ with } Z_h(0) = 0. \quad (4.5)$$

We will use Z_α as the shorthand of Z_h when $h_s = \int_0^s (\tilde{T}_r^{-1})^* e_\alpha dr$, $1 \leq \alpha \leq d$.

Lemma 4.15 Let X^{Z_α} be given above, then $Z_\alpha \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$.

Proof. Notice that Z_α satisfies the following ODE:

$$Z'_\alpha(s) = -\frac{1}{2}Ric_{\tilde{u}_s}Z_\alpha(s) + (\tilde{T}_s^{-1})^* e_\alpha \text{ with } Z_\alpha(0) = 0. \quad (4.6)$$

Since $(\tilde{T}_s^{-1})^* e_\alpha$ is adapted, Z'_α is adapted. So Z_α is a Brownian semimartingale with $Q \equiv 0$ and $r = Z'_\alpha$. Since \tilde{T}_s is bounded, from Gronwall's inequality we have Z_α is bounded $\nu - a.s$, and the bound is independent of $\sigma \in W_o(M)$ and $s \in [0, 1]$. Therefore $Z_\alpha \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$. ■

Definition 4.16 Define the domain of \tilde{X} to be

$$\mathcal{D}(\tilde{X}) := \cap_{\alpha=1}^d \mathcal{D}(X^{Z_\alpha})$$

and for any $f \in \mathcal{D}(\tilde{X})$, set

$$\tilde{X}f := \sum_{\alpha=1}^d \langle \tilde{C}\tilde{H}, e_\alpha \rangle X^{Z_\alpha}f,$$

where $\tilde{C} = \left[\int_0^1 (\tilde{T}_r^* \tilde{T}_r)^{-1} dr \right]^{-1} \tilde{T}_1^{-1}$.

Remark 4.17 To motivate this definition, we formally use the H -derivative. Notice that from Definition 3.19:

$$\Phi_s = \int_0^s (\tilde{T}_\tau^{-1})^* \left[\int_0^1 (\tilde{T}_r^* \tilde{T}_r)^{-1} dr \right]^{-1} \tilde{T}_1^{-1} \tilde{H} d\tau = \sum_{\alpha=1}^d \langle \tilde{C}\tilde{H}, e_\alpha \rangle \int_0^s (\tilde{T}_r^{-1})^* e_\alpha dr,$$

by superposition principle,

$$Z_\Phi(s) = \sum_{\alpha=1}^d \langle \tilde{C}\tilde{H}, e_\alpha \rangle Z_\alpha(s)$$

and further

$$X^{Z_\Phi}f = \langle Df, X^{Z_\Phi} \rangle_{G^1} = \sum_{\alpha=1}^d \langle \tilde{C}\tilde{H}, e_\alpha \rangle \langle Df, X^{Z_\alpha} \rangle_{G^1} = \sum_{\alpha=1}^d \langle \tilde{C}\tilde{H}, e_\alpha \rangle X^{Z_\alpha}f. \quad (4.7)$$

4.2 Computing $\tilde{X}^{tr, \nu}$

This subsection is devoted to the study of $\tilde{X}^{tr, \nu}$ (The adjoint operator of \tilde{X} with respect to ν restricted to $\mathcal{D}(\tilde{X})$). The crucial step to show its existence is checking the anticipating coefficients in (4.7) are differentiable in the sense of Definition 4.8.

Proposition 4.18 Our standard assumption of bounded curvature tensor implies that Ric is bounded. If we further assume ∇R is bounded, then for any $h \in \mathcal{V}^{\infty-} \cap \mathcal{B}^{\infty-}$ and $s \in [0, 1]$, we have $Ric_{\tilde{u}_s} \in \mathcal{D}(X^h)$. Moreover, the map

$$s \rightarrow X^h Ric_{\tilde{u}_s} \in S^{\infty-}. \quad (4.8)$$

Proof. Since for any $s \in [0, 1]$, $Ric_{\tilde{u}_s} \in \mathcal{GF}\mathcal{C}^1$, from Theorem 4.13 we know $Ric_{\tilde{u}_s} \in \mathcal{D}(X^h)$ and

$$X^h Ric_{\tilde{u}_s} = (\nabla_{X_s^h} Ric)_{\tilde{u}_s} + [A_s \langle h \rangle, Ric_{\tilde{u}_s}], \quad (4.9)$$

where $[\cdot, \cdot]$ is the Lie bracket of matrices and $(\nabla_{X_s^h} Ric)_{\tilde{u}_s} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined to be

$$(\nabla_{X_s^h} Ric)_{\tilde{u}_s} = \tilde{u}_s^{-1} (\nabla_{X_s^h} Ric) \cdot \tilde{u}_s.$$

Since ∇Ric is bounded,

$$\left| (\nabla_{X_s^h} Ric)_{\tilde{u}_s} \right| \leq C \langle X_s^h, X_s^h \rangle_g^{\frac{1}{2}} = C |h_s| \leq Ch^*,$$

where C is a constant and h^* is the essential supremum of $s \rightarrow h_s$. For any $q \in [1, \infty)$, since $h \in \mathcal{B}^{\infty-} \subset S^{\infty-}$, we know

$$\sup_{s \in [0, 1]} \left| (\nabla_{X_s^h} Ric)_{\tilde{u}_s} \right| \in L^{\infty-}(W_o(M)). \quad (4.10)$$

Then we express $A_s \langle h \rangle$ in Itô form:

$$A_s \langle h \rangle = \int_0^s R_{\tilde{u}_r} (h_r, d\beta_r) + \frac{1}{2} \sum_{i=1}^d \int_0^s \left\{ R_{\tilde{u}_r} (Q_r^h e_i, e_i) + \left(\frac{d}{dt} \Big|_0 R_{e^{tB_{e_i}}(\tilde{u}_r)} \right) (h_r, e_i) \right\} dr.$$

Since R and ∇R are bounded, for any $s \in [0, 1], q \geq 1$,

$$\left| \frac{1}{2} \sum_{i=1}^d \int_0^s \left\{ R_{\tilde{u}_r} (Q_r^h e_i, e_i) + \left(\frac{d}{dt} \Big|_0 R_{e^{tB_{e_i}}(\tilde{u}_r)} \right) (h_r, e_i) \right\} dr \right|^q \leq C_q \left(\|Q^h\|_{L^q([0, 1])}^q + (h^*)^q \right). \quad (4.11)$$

Using Burkholder-Davis-Gundy inequality, for any $q \in [1, \infty)$,

$$\mathbb{E} \left[\sup_{s \in [0, 1]} \left| \int_0^s R_{\tilde{u}_r} (h_r, d\beta_r) \right|^q \right] \leq C \|h\|_{L^{\frac{q}{2}}(W_o(M))}^{\frac{q}{2}} < \infty. \quad (4.12)$$

Combining Eq.(4.11) and (4.12) we have

$$\sup_{s \in [0, 1]} |A_s \langle h \rangle| \in L^{\infty-}(W_o(M)).$$

Since Ric is bounded, we have

$$\sup_{s \in [0, 1]} |[A_s \langle h \rangle, Ric_{\tilde{u}_s}]| \in L^{\infty-}(W_o(M)). \quad (4.13)$$

Combining (4.9), (4.10) and (4.13) gives (4.8). ■

Lemma 4.19 Let \tilde{C} be as defined in Lemma 4.16, then $\tilde{C} \in L^{\infty-}(W_o(M), \nu)$.

Proof. Since $\|\tilde{T}_1^{-1}\|$ is bounded $\nu - a.s.$, it suffices to show $\left\| \left(\int_0^1 \tilde{T}_r^{-1} (T_r^{-1})^* dr \right)^{-1} \right\|$ is bounded $\nu - a.s.$ For any $v \in \mathbb{C}^d$,

$$\left\langle \left(\int_0^1 \tilde{T}_r^{-1} (\tilde{T}_r^{-1})^* dr \right) v, v \right\rangle = \int_0^1 \|(\tilde{T}_r^{-1})^* v\|^2 dr \geq C \|v\|^2. \nu - a.s.$$

So

$$\left\| \left(\int_0^1 \tilde{T}_r^{-1} (T_r^{-1})^* dr \right)^{-1} \right\| \leq \frac{1}{C} \nu - a.s.$$

where C is a deterministic constant ■

Theorem 4.20 Let \tilde{T}_s be as defined in Definition 3.14, then $\tilde{T}_s \in \mathcal{D}(X^{Z_\alpha})$ for $1 \leq \alpha \leq d$.

First we state a supplementary lemma.

Lemma 4.21 Let $\{f_{(\cdot)}(t)\}_{t \in \mathbb{R}}$ be V -valued Brownian semi-martingales which are \mathcal{B}^p -differentiable for some $p \geq 1$ at $t = 0$, then for any $s \in [0, 1]$, $\{f_s(t)\}_{t \in \mathbb{R}}$ are differentiable at $t = 0$ in $L^p(W_o(M) \rightarrow V)$. Furthermore,

$$\left\| \frac{f_s(t) - f_s(0)}{t} - \frac{d}{dt} \Big|_0 f_s(t) \right\|_{L^p(W_o(M) \rightarrow V)} \rightarrow 0 \text{ as } t \rightarrow 0 \text{ uniformly with respect to } s.$$

Proof. We represent $f_{(\cdot)}(t) := \int_0^t Q_s^f(t) d\beta_s + \int_0^t r_s^f(t) ds$ and denote $\frac{d}{dt} \Big|_0 f_{(\cdot)}(t)$ by

$$g_{(\cdot)} := \int_0^t Q_s^g d\beta_s + \int_0^t r_s^g ds.$$

$\frac{d}{dt} \Big|_0 f_{(\cdot)}(t) = g_{(\cdot)}$ in \mathcal{B}^p implies that $\frac{d}{dt} \Big|_0 Q^f(t) = Q^g$ in $S^p(\text{Hom}(\mathbb{R}^d, V))$ and $\frac{d}{dt} \Big|_0 r^f(t) = r^g$ in $S^p(V)$. Since for any $s \in [0, 1]$,

$$\begin{aligned} & \left\| \frac{f_s(t) - f_s(0)}{t} - g_s \right\|_V^p \\ & \leq C_p \left[\left\| \int_0^s \left(\frac{Q_\tau^f(t) - Q_\tau^f(0)}{t} - Q_\tau^g \right) d\beta_\tau \right\|_V^p + \left\| \int_0^s \left(\frac{r_\tau^f(t) - r_\tau^f(0)}{t} - r_\tau^g \right) d\tau \right\|_V^p \right] \end{aligned}$$

Taking expectation on both hand side and using Burkholder-Davis-Gundy inequality on the first term, we have

$$\left\| \frac{f_s(t) - f_s(0)}{t} - g_s \right\|_{L^p(W_o(M) \rightarrow V)} \leq C_p \left[\left\| \frac{Q_\tau^f(t) - Q_\tau^f(0)}{t} - Q_\tau^g \right\|_{S^p} + \left\| \frac{r_\tau^f(t) - r_\tau^f(0)}{t} - r_\tau^g \right\|_{S^p} \right] \rightarrow 0$$

as $t \rightarrow 0$. The uniformity with respect to s is easily seen from the fact that the dominating function is independent of s . ■

Proof of Theorem 4.20. For each X^{Z_α} , since $Z_\alpha \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$ by Lemma 4.15, we can construct an approximate flow $E(tX^{Z_\alpha})$ of X^{Z_α} . Define $\tilde{T}_s(t) := \tilde{T}_s \circ E(tX^{Z_\alpha})$ and $G_s(t) := \frac{\tilde{T}_s(t) - \tilde{T}_s}{t}$, it is easy to see that $G_s(t)$ satisfies the following ODE:

$$G'_s(t) = -\frac{1}{2} Ric_{\tilde{u}_s} G_s(t) - \frac{1}{2t} (Ric_{\tilde{u}_s(t)} - Ric_{\tilde{u}_s}) \tilde{T}_s(t) \text{ with } G_0(t) = 0,$$

where $\tilde{u}_{(\cdot)}(t)$ is the stochastic parallel translation along $E(tX^{Z_\alpha})$ and "r" is the derivative with respect to parameter s .

Then denote by G_s the solution to the following ODE

$$G'_s = -\frac{1}{2} Ric_{\tilde{u}_s} G_s - \frac{1}{2} (X^{Z_\alpha} Ric_{\tilde{u}_s}) \tilde{T}_s \text{ with } G_0 = 0$$

and let $H_s(t)$ be $G_s(t) - G_s$. We know $H_s(t)$ satisfies

$$H'_s(t) = -\frac{1}{2} Ric_{\tilde{u}_s} H_s(t) - \frac{1}{2} \left(\frac{Ric_{\tilde{u}_s(t)} - Ric_{\tilde{u}_s}}{t} \tilde{T}_s(t) + (X^{Z_\alpha} Ric_{\tilde{u}_s}) \tilde{T}_s \right), H_0(t) = 0.$$

According to Definition 4.8,

$$\tilde{T}_s \in \mathcal{D}(X^{Z_\alpha}) \iff H_s(t) \rightarrow 0 \text{ in } L^{\infty-}(W_o(M)).$$

By Gronwall's inequality, we have

$$|H_s(t)| \leq \int_0^s \left| \frac{Ric_{\tilde{u}_r(t)} - Ric_{\tilde{u}_r}}{t} \tilde{T}_r(t) + (X^{Z_\alpha} Ric_{\tilde{u}_r}) \tilde{T}_r \right| dr e^{\frac{d(N-1)}{2}}.$$

Following Theorem 4.13 we know for any $p \geq 1$, $r \in [0, 1]$,

$$\frac{Ric_{\tilde{u}_r(t)} - Ric_{\tilde{u}_r}}{t} \rightarrow X^{Z_\alpha} Ric_{\tilde{u}_r} \text{ as } t \rightarrow 0 \text{ in } L^p(W_o(M)). \quad (4.14)$$

Since Ric , ∇Ric are bounded and $Z_\alpha \in \mathcal{V}^\infty \cap \mathcal{B}^\infty$, Lemma 4.21 shows that this convergence is uniform with respect to $r \in [0, 1]$. Since $\sup_{0 \leq r \leq 1} \|\tilde{T}_r\|$ is bounded, using bounded convergence theorem, we have

$$\tilde{T}_r(t) \rightarrow \tilde{T}_r \text{ in } L^{\infty-}(W_o(M)) \text{ uniformly with respect to } r \in [0, 1]. \quad (4.15)$$

Combining (4.14) and (4.15) we have $H_s(t) \rightarrow 0$ in $L^{\infty-}(W_o(M))$ as $t \rightarrow 0$. ■

Corollary 4.22 *Recall that we have defined $\tilde{C} = \left[\int_0^1 \left(\tilde{T}_r^* \tilde{T}_r \right)^{-1} dr \right]^{-1} \tilde{T}_1^{-1}$ in Lemma 4.16, then*

$$\tilde{C} \in \mathcal{D}(X^{Z_\alpha}) \text{ for } 1 \leq \alpha \leq d.$$

Proof. Lemma 4.19 shows that $\tilde{C} \in L^{\infty-}(W_o(M))$. By the product rule and Theorem 4.20, for any $s \in [0, 1]$,

$$X^{Z_\alpha} \left(\tilde{T}_s^{-1} \right) = -\tilde{T}_s \left(X^{Z_\alpha} \tilde{T}_s \right) \tilde{T}_s \in L^{\infty-}(W_o(M)),$$

so $\tilde{T}_s^{-1} \in \mathcal{D}(X^{Z_\alpha})$ and thus $\int_0^1 \left(\tilde{T}_r^* \tilde{T}_r \right)^{-1} dr \in \mathcal{D}(X^{Z_\alpha})$. Then apply the product rule again we get $\tilde{C} \in \mathcal{D}(X^{Z_\alpha})$. ■

Lemma 4.23 *Given $X \in \Gamma(TM)$ with compact support, if \tilde{X} is its orthogonal lift on $W_o(M)$, then define an operator on $L^2(W_o(M), \nu)$ by*

$$\tilde{X}^{tr, \nu} = -\tilde{X} + \sum_{\alpha=1}^d \left\langle \tilde{C} \tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1} \right)^* e_\alpha, d\beta_s \right\rangle + \sum_{\alpha=1}^d \left\langle -X^{Z_\alpha} \left(\tilde{C} \tilde{H} \right), e_\alpha \right\rangle$$

with $\mathcal{D}(\tilde{X}^{tr, \nu}) := \mathcal{D}(\tilde{X})$, then for any $f, g \in \mathcal{D}(\tilde{X})$, we have

$$\mathbb{E}_\nu \left[\tilde{X} f \cdot g \right] = \mathbb{E}_\nu \left[f \cdot \tilde{X}^{tr, \nu} g \right].$$

Proof. Since $\tilde{H} \in \mathcal{GFC}^1$, $\tilde{H} \in \mathcal{D}(X^{Z_\alpha}) \forall 1 \leq \alpha \leq d$. Based on this observation and Corollary 4.22, we obtain

$$\mathbb{E} \left[\tilde{X} f \cdot g \right] = \mathbb{E} \left[\sum_{\alpha=1}^d \left\langle \tilde{C} \tilde{H}, e_\alpha \right\rangle X^{Z_\alpha} f \cdot g \right] = \sum_{\alpha=1}^d \mathbb{E} \left[X^{Z_\alpha} f \cdot \left(g \cdot \left\langle \tilde{C} \tilde{H}, e_\alpha \right\rangle \right) \right] = I + II + III \quad (4.16)$$

where

$$\begin{aligned} I &= \mathbb{E} \left[f \cdot \left(-\tilde{X} \right) g \right] \\ II &= \mathbb{E} \left[f \cdot g \cdot \sum_{\alpha=1}^d \left\langle \tilde{C} \tilde{H}, e_\alpha \right\rangle \int_0^1 \left\langle \left(\tilde{T}_s^{-1} \right)^* e_\alpha, d\beta_s \right\rangle \right] \\ III &= \mathbb{E} \left[f \cdot g \cdot \sum_{\alpha=1}^d \left\langle -X^{Z_\alpha} \left(\tilde{C} \tilde{H} \right), e_\alpha \right\rangle \right]. \end{aligned}$$

Since $f \in L^{\infty-}(W_o(M), \nu)$, the proof can be completed by showing $\tilde{X}^{tr, \nu} g \in L^{\infty-}(W_o(M), \nu)$. Corollary 4.22 and the fact that $\tilde{H} \in \mathcal{D}(X^{Z_\alpha})$ implies that $\tilde{C} \tilde{H}, X^{Z_\alpha} \left(\tilde{C} \tilde{H} \right) \in L^{\infty-}(W_o(M), \nu)$, so

it suffices to show $\int_0^1 \left\langle \left(\tilde{T}_s^{-1} \right)^* e_\alpha, d\beta_s \right\rangle \in L^{\infty-}(W_o(M), \nu)$. The fact that it is true is a result of the boundedness of $\sup_{0 \leq s \leq 1} \left\| \tilde{T}_s^{-1} \right\|$ and Burkholder-Davis-Gundy inequality.

■

The following lemma gives a more explicit expression of the last term in $\tilde{X}^{tr, \nu}$

$$\sum_{\alpha=1}^d \left\langle -X^{Z_\alpha} \left(\tilde{C} \tilde{H} \right), e_\alpha \right\rangle$$

under an extra condition that $\nabla R \equiv 0$. The new expression indicates a structure of the divergence term $\tilde{X}^{tr, \nu}$ that is analogous to finite dimensional Riemannian geometry. Interested reader may refer to the structure theory on Appendix B.

Lemma 4.24 *If further curvature tensor is parallel, i.e. $\nabla R \equiv 0$, then*

$$-\sum_{\alpha=1}^d \left\langle X^{Z_\alpha} \left(\tilde{C} \tilde{H} \right), e_\alpha \right\rangle = \text{div} X \circ E_1 - \sum_{\alpha=1}^d \left\langle \tilde{C} A_1 \langle Z_\alpha \rangle \tilde{H}, e_\alpha \right\rangle. \quad (4.17)$$

Proof. Since for tensors, contraction commutes with covariant differentiation, and Ric is the contraction of curvature tensor R , so $\nabla Ric \equiv 0$ and thus $\delta Ric_{\tilde{u}_s} = \nabla_{\delta \beta_s} Ric \equiv 0$. So $Ric_{\tilde{u}_s} = Ric_{\tilde{u}_0}$ a.s. and it follows that \tilde{T}_s and \tilde{C} have deterministic versions.

Since $\tilde{H} = \tilde{u}_1^{-1} X (\pi \circ \tilde{u}_1) \in \mathcal{GFC}^1$, we can apply Theorem 4.13 to \tilde{H} to find

$$\sum_{\alpha=1}^d \left\langle X^{Z_\alpha} \left(\tilde{C} \tilde{H} \right), e_\alpha \right\rangle = \sum_{\alpha=1}^d \left\langle \tilde{C} X^{Z_\alpha} \tilde{H}, e_\alpha \right\rangle = I + II$$

where

$$I = -\sum_{\alpha=1}^d \left\langle \tilde{C} \tilde{u}_1^{-1} \nabla_{X^{Z_\alpha(1)}} X, e_\alpha \right\rangle \quad \text{and} \quad II = \sum_{\alpha=1}^d \left\langle \tilde{C} A_1 \langle Z_\alpha \rangle \tilde{H}, e_\alpha \right\rangle.$$

Claim: $I = -\text{div} X \circ E_1$.

Proof of Claim:

$$I = -\sum_{\alpha=1}^d \left\langle \tilde{u}_1 \tilde{C} \tilde{u}_1^{-1} \nabla_{\tilde{u}_1 \tilde{C}^{-1} \tilde{u}_1^{-1} e_\alpha} X, \tilde{u}_1 e_\alpha \right\rangle = -\sum_{\alpha=1}^d \left\langle A^{-1} \nabla_{A f_\alpha} X, f_\alpha \right\rangle = -\sum_{\alpha=1}^d \left\langle \nabla_{A f_\alpha} X, (A^{-1})^* f_\alpha \right\rangle$$

where $A = \tilde{u}_1 \tilde{C}^{-1} \tilde{u}_1^{-1} \in \text{End}(T_{E_1(\sigma)} M)$ and $\{f_\alpha\} = \{\tilde{u}_1 e_\alpha\}$ is an orthonormal basis of $T_{E_1(\sigma)} M$. Since $\langle \nabla \cdot X, \cdot \rangle$ is bilinear on $T_{E_1(\sigma)} M$, by the universal property of tensor product we know there exists a linear map $l : T_{E_1(\sigma)} M \otimes T_{E_1(\sigma)} M \mapsto \mathbb{R}$ such that

$$\left\langle \nabla_{A f_\alpha} X, (A^{-1})^* f_\alpha \right\rangle = l \left(A f_\alpha \otimes (A^{-1})^* f_\alpha \right)$$

and therefore:

$$\sum_{\alpha=1}^d \left\langle \nabla_{A f_\alpha} X, (A^{-1})^* f_\alpha \right\rangle = l \left(\sum_{\alpha=1}^d A f_\alpha \otimes (A^{-1})^* f_\alpha \right). \quad (4.18)$$

Using the isomorphism between $T^{1,1}(V) \mapsto \text{End}(V) : (a \otimes b) v = a \cdot \langle b, v \rangle$ one can easily see:

$$\sum_{\alpha=1}^d A f_\alpha \otimes (A^{-1})^* f_\alpha = \sum_{\alpha=1}^d f_\alpha \otimes f_\alpha. \quad (4.19)$$

Combining (4.18) and (4.19) we have

$$I = -\sum_{\alpha=1}^d \left\langle \nabla_{f_\alpha} X, f_\alpha \right\rangle = -\text{div} X \circ E_1$$

and thus (4.17). ■

A ODE estimates

Lemma A.1 *Let $\alpha(t) \in \text{End}(\mathbb{R}^d)$ be a continuously varying matrix valued function and $S(t) \in \text{End}(\mathbb{R}^d)$ be the solution to the following initial value problem:*

$$\frac{d}{dt}S(t) = \alpha(t)S(t), \quad S(0) = I,$$

then for any $t \in [0, 1]$, $S(t) \in \text{Aut}(\mathbb{R}^d)$. Furthermore,

$$\int_0^t [S(r)^* S(r)]^{-1} dr \in \text{Aut}(\mathbb{R}^d) \quad \forall t \in [0, 1].$$

Proof. Denote by $U(t) \in \text{End}(\mathbb{R}^d)$ the solution to the following initial value problem:

$$\frac{d}{dt}U(t) = -U(t)\alpha(t), \quad U(0) = I,$$

then direct computation shows that $Y(t) := S(t)U(t) \in \text{End}(\mathbb{R}^d)$ satisfies

$$\frac{d}{dt}Y(t) = \alpha(t)Y(t) - Y(t)\alpha(t), \quad Y(0) = I.$$

By the uniqueness of solutions for linear ODE, we get $S(t)U(t) \equiv I$, and this shows that $U(t)$ is a left inverse to $S(t)$. As we are in finite dimensions it follows that $T(t)^{-1}$ exists and is equal to $U(t)$. Then for any $v \in \mathbb{C}^d / \{0\}$,

$$\left\langle \int_0^t [S(r)^* S(r)]^{-1} dr v, v \right\rangle = \int_0^t \langle [S(r)^*]^{-1} v, [S(r)^*]^{-1} v \rangle dr = \int_0^t \|U(r)v\|^2 dr \quad (\text{A.1})$$

Since $U(0) = I$ and $U(\cdot) : [0, 1] \rightarrow \text{Aut}(\mathbb{R}^d)$ is continuous,

$$\left\langle \int_0^t [S(r)^* S(r)]^{-1} dr v, v \right\rangle > 0$$

and this implies $\int_0^t [S(r)^* S(r)]^{-1} dr \in \text{Aut}(\mathbb{R}^d) \quad \forall t \in [0, 1]$. ■

B A Structure Theorem for $\text{div}_g(\tilde{X})$

This section is devoted to a structure theorem for $\text{div}_g(\tilde{X})$ —the divergence of the lifted vector field \tilde{X} in finite dimensional Riemannian geometry. We expect that the orthogonal lift that we introduced in this paper also has an analogous structure, as is hinted in Lemma 4.24.

Let $\pi : (M, g) \rightarrow (N, h)$ be a submersion of two smooth Riemannian manifolds. To each $m \in M$ and $v \in T_{\pi(m)}N$, let $\hat{v} := \pi_{*m}^{\text{tr}} (\pi_{*m} \pi_{*m}^{\text{tr}})^{-1} v \in T_m M$ so that \hat{v} is the unique shortest vector in $T_m M$ such that $\pi_{*m} \hat{v} = v$. So if $X \in \Gamma(TN)$ is a vector field on N , then $\hat{X} \in \Gamma(TM)$ is defined by $\hat{X}(m) = \pi_{*m}^{\text{tr}} (\pi_{*m} \pi_{*m}^{\text{tr}})^{-1} X(\pi(m))$ and we have $\pi_* \hat{X} = X \circ \pi$. Finally, let Vol_g and Vol_h be the volume forms on (M, g) and (N, h) respectively.

Lemma B.1 *If $K := \dim M > k := \dim N$, then there exists a unique $K - k$ - form (γ) on M such that;*

1. $\text{Vol}_g = (\pi^* \text{Vol}_h) \wedge \gamma$
2. $i_{\hat{v}} \gamma = 0$ for any $v \in T_{\pi(m)}N$ and $m \in M$.

Proof. Uniqueness. Assuming such a γ exists, choose an orthonormal basis $\{e_1, \dots, e_k\}$ for $T_{\pi(m)}N$ such that $\text{Vol}_h(e_1, \dots, e_k) = 1$. Then it follows that

$$\begin{aligned} \text{Vol}_g(\hat{e}_1, \dots, \hat{e}_k, \cdot, \dots, \cdot) &= (\pi^* \text{Vol}_h)(\hat{e}_1, \dots, \hat{e}_k) \wedge \gamma \\ &= \text{Vol}_h(\pi_* \hat{e}_1, \dots, \pi_* \hat{e}_k) \wedge \gamma \\ &= \text{Vol}_h(e_1, \dots, e_k) \wedge \gamma = \gamma \end{aligned}$$

which shows γ is unique if it exists.

Existence. Now suppose that $\{e_1, \dots, e_k\}$ is a local orthonormal frame on M in a neighborhood of $\pi(m)$ such that $\text{Vol}_h(e_1, \dots, e_k) = 1$. Then by above we must define

$$\gamma := \text{Vol}_g(\hat{e}_1, \dots, \hat{e}_k, \cdot, \dots, \cdot) \text{ in a neighborhood of } m.$$

It is now straightforward to check that this γ has the desired properties and is defined independent of the choice of frame. ■

Corollary B.2 *If $X \in \Gamma(TN)$ and $\hat{X} \in \Gamma(TM)$ is its lift as described above, then*

$$\text{div}_g(\hat{X}) = \text{div}_h(X) \circ \pi + \rho_{\hat{X}}$$

where $\rho_{\hat{X}}(m)$ is a function on M depending only on $\hat{X}(m)$. {To compute $\rho_{\hat{X}}$ explicitly will require a better understanding of $d\gamma$.}

Proof. From Lemma B.1 we learn,

$$\begin{aligned} \text{div}_g(\hat{X}) \text{Vol}_g &= d[i_{\hat{X}} \text{Vol}_g] = d[i_{\hat{X}}((\pi^* \text{Vol}_h) \wedge \gamma)] \\ &= d[(i_{\hat{X}}(\pi^* \text{Vol}_h) \wedge \gamma)] \\ &= [d(i_{\hat{X}}(\pi^* \text{Vol}_h))] \wedge \gamma + (-1)^k (i_{\hat{X}}(\pi^* \text{Vol}_h) \wedge d\gamma). \end{aligned}$$

Since

$$\begin{aligned} i_{\hat{X}}(\pi^* \text{Vol}_h) &= (\pi^* \text{Vol}_h)(\hat{X}, \dots) = \text{Vol}_h(\pi_* \hat{X}, \pi_* \dots) \\ &= \text{Vol}_h(X \circ \pi, \pi_* \dots) = \pi^*(i_X \text{Vol}_h) \end{aligned}$$

it follows that

$$\begin{aligned} d(i_{\hat{X}}(\pi^* \text{Vol}_h)) &= d(\pi^*(i_X \text{Vol}_h)) = \pi^*(d(i_X \text{Vol}_h)) \\ &= \pi^*(\text{div}_h(X) \text{Vol}_h) = \text{div}_h(X) \circ \pi \cdot \pi^* \text{Vol}_h. \end{aligned}$$

Combining these equations then shows,

$$\begin{aligned} \text{div}_g(\hat{X}) \text{Vol}_g &= \text{div}_h(X) \circ \pi \cdot (\pi^* \text{Vol}_h) \wedge \gamma + (-1)^k (i_{\hat{X}}(\pi^* \text{Vol}_h) \wedge d\gamma) \\ &= [\text{div}_h(X) \circ \pi + \rho_{\hat{X}}] \cdot \text{Vol}_g \end{aligned}$$

where

$$\rho_{\hat{X}} = \frac{(-1)^k (i_{\hat{X}}(\pi^* \text{Vol}_h) \wedge d\gamma)}{\text{Vol}_g}.$$

■

References

- [1] Shigeki Aida, *On the irreducibility of certain Dirichlet forms on loop spaces over compact homogeneous spaces*, New trends in stochastic analysis (Charingworth, 1994), World Sci. Publ., River Edge, NJ, 1997, pp. 3–42. MR 1654499 1.1
- [2] Lars Andersson and Bruce K. Driver, *Finite-dimensional approximations to Wiener measure and path integral formulas on manifolds*, J. Funct. Anal. **165** (1999), no. 2, 430–498. MR 2000j:58059 2
- [3] Denis Bell, *Divergence theorems in path space*, J. Funct. Anal. **218** (2005), no. 1, 130–149. MR 2101217 1.1
- [4] R. H. Cameron and W. T. Martin, *Transformations of Wiener integrals under translations*, Ann. of Math. (2) **45** (1944), 386–396. MR 0010346 (6,5f) 1.1
- [5] Ana Bela Cruzeiro and Shizan Fang, *Weak Levi-Civita connection for the damped metric on the Riemannian path space and vanishing of Ricci tensor in adapted differential geometry*, J. Funct. Anal. **185** (2001), no. 2, 681–698. MR 1856279 1.1
- [6] Ana-Bela Cruzeiro and Paul Malliavin, *Renormalized differential geometry on path space: structural equation, curvature*, J. Funct. Anal. **139** (1996), no. 1, 119–181. MR 1399688 (97h:58175) 1.1, 2
- [7] B. K. Driver, *The Lie bracket of adapted vector fields on Wiener spaces*, Appl. Math. Optim. **39** (1999), no. 2, 179–210. MR 2000b:58063 4.1, 4.1, 4.1
- [8] Bruce K. Driver, *A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold*, J. Funct. Anal. **110** (1992), no. 2, 272–376. 1.1, 2, 2.19
- [9] ———, *A Cameron-Martin type quasi-invariance theorem for pinned Brownian motion on a compact Riemannian manifold*, Trans. Amer. Math. Soc. **342** (1994), no. 1, 375–395. 1.1
- [10] ———, *A primer on riemannian geometry and stochastic analysis on path spaces*, ETH (Zürich, Switzerland) preprint series. This may be retrieved at <http://math.ucsd.edu/~driver/prgsaps.html>, 1995. 2
- [11] ———, *Towards calculus and geometry on path spaces*, Stochastic analysis (Ithaca, NY, 1993), Proc. Sympos. Pure Math., vol. 57, Amer. Math. Soc., Providence, RI, 1995, pp. 405–422. 1.1
- [12] K. D. Elworthy, *Stochastic differential equations on manifolds*, London Mathematical Society Lecture Note Series, vol. 70, Cambridge University Press, Cambridge-New York, 1982. MR 675100 2
- [13] K. David Elworthy and Xue-Mei Li, *Geometric stochastic analysis on path spaces*, International Congress of Mathematicians. Vol. III, Eur. Math. Soc., Zürich, 2006, pp. 575–594. MR 2275697 1.1
- [14] Ognian Enchev and Daniel W. Stroock, *Towards a Riemannian geometry on the path space over a Riemannian manifold*, J. Funct. Anal. **134** (1995), no. 2, 392–416. MR 1363806 (96m:58270) 1.1
- [15] Elton P. Hsu, *Quasi-invariance of the Wiener measure on the path space over a compact Riemannian manifold*, J. Funct. Anal. **134** (1995), no. 2, 417–450. MR 1363807 (97c:58163) 1.1
- [16] ———, *Quasi-invariance of the Wiener measure on path spaces: noncompact case*, J. Funct. Anal. **193** (2002), no. 2, 278–290. MR 1929503 (2003i:58069) 1.1
- [17] ———, *Stochastic analysis on manifolds*, Graduate Studies in Mathematics, vol. 38, American Mathematical Society, Providence, RI, 2002. MR 1882015 2, 2

- [18] Elton P. Hsu and Cheng Ouyang, *Quasi-invariance of the Wiener measure on the path space over a complete Riemannian manifold*, J. Funct. Anal. **257** (2009), no. 5, 1379–1395. MR 2541273 (2010h:58054) 1.1
- [19] Wilhelm Klingenberg, *Lectures on closed geodesics*, third ed., Mathematisches Institut der Universität Bonn, Bonn, 1977. MR 0461361 1.2, 1.7